SHAPE DESIGN SENSITIVITY ANALYSIS BASED ON BOUNDARY INTEGRAL EQUATION METHOD CONSIDERING GENERAL SHAPE VARIATIONS

PART I : FOR SELF-ADJOINT ELLIPTIC OPERATOR PROBLEMS

Byung Man Kwak* and **Joo Ho Choi***

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A procedure for shape design sensitivity analysis is developed using a standard boundary integral equation (BIE) formulation for elliptic problems with static response. The performance functional to be considered involves both the domain and boundary integrals, and a complete consideration is given in describing the shape variation by including the tangential as well as the normal component of the velocity field. The material derivative concept and the adjoint variable method as applied to the BIE formulation are basic tools for the derivation. This has opened a new unified approach using the BIE to the shape design sensitivity problems.

Key Words : Shape Design Sensitivity Analysis, Boundary Integral Equation(BIE), Material Derivative, Adjoint Variable Method

1. INTRODUCTION

Shape optimal design is an important class of design problems in which the shape of a problem domain is to be determined, such that a cost functional is minimized under various constraints. This topic has been studied in various fields such as optimal control, calculus of variation and identification, mainly in an applied mathematical aspect. With the advent of high speed computers, new study, especially in computational or applicational aspect, has been activated recently. A recent symposium(Haug, 1981) has devoted mainly to this topic, and great attention has followed as can be seen in recent literatures (Pironneau, 1984), (Haug, 1985) and (Mota Soares, 1986).

The problem can be described as follows:

Find the shape of Ω such that

minimize $\Phi_0(u\,;\,\Omega)$ (1)

subject to $\Phi_i(u\,;\,\Omega)\leq 0,\ i=1,2,\cdots,$ (2)

and $\Lambda(u\,;\,\Omega)=0,$ (3)

where the problem domain which is to be determined is represented by Ω, and u denotes the state function which can be obtained from the state Eq. (3), once Ω is specified.

The objective of this paper is to develop a calculational procedure for shape optimization based on boundary integral equation formulation. The key step in the numerical optimization utilizing a gradient type method is to find the gradient or the sensitivity of the functional Φ, that is, the change of Φ with respect to an infinitesimal change in the shape of Ω. Many calculational methods have been introduced mainly with a differential equation form or with an algebraic equation form after discretization. In fact, most of the existing works on the numerical shape optimization have been done by the finite element methods, based on a variational approach (Haftka, 1986). In this approach, a significant drawback is inherent inaccuracy in the calculation of boundary variables, which always appear in the sensitivity expressions. In view of this, the boundary element method which can generate more accurate values on the boundary, provides a good alternative for better derivative calculations.

In a previous work by one of the author's of the present paper(Choi, 1987a), a procedure was developed for the variation of performance functional represented as boundary integrals, in terms of normal movement of the boundary. The present paper presents some generalization : A domain integral is included in the functional, and a full variation of shape including both the normal and tangential component of the velocity field is considered. In the following development, we restrict ourselves to the case of sufficiently smooth boundary and smooth functions in the problem formulation.

There have been a sequence of papers on shape design sensitivity analysis by(Choi, 1983) and (Dems, 1984), et. al., in which variational formulations are mainly utilized. While the origins of the method presented herein and those are quite different, consistent results as well as complementary ones are obtained. The relationship of the present formulas and those by(Choi, 1983) will also be noted, when concrete examples are considered.

In Section 2, a formal boundary integral equation(BIE) is derived for rather general elliptic boundary value problems with static response, and a variation of functional including both domain as well as boundary integrals is then presented in Section 3, obtaining a comprehensive explicit formula for the sensitivity. The material derivative concept and the adjoint variable method (Haug, 1985) are the basic tools in the derivation. Concrete formulas corresponding to the general formula obtained here, are given in the next paper(Choi, 1987b), specially for potential, elasticity and plate bending problems. Some numerical applications to practical problems of shape optimization can be found in(Kwak, 1986) and (Kwak, 1987) for a seepage and elastic component optimization problems.

2. FORMULATION OF FORMAL BIE

Let Ω be an open domain in $\mathbf{R}^n$ with a sufficiently smooth boundary $\partial\Omega$. we wish to consider a class of linear, self-adjoint, elliptic differential operator A with order 2m defined in Ω. A system of boundary operators $\{B_i\}_{i=0}^{m-1}$ with order i and $\{C_i\}_{i=0}^{m-1}$ with order m+i, associated with A and Ω,

* Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, Seoul 131, Korea

are defined as the Dirichlet and Neumann operator, respectively. Then, for a proper system of boundary operators, the following generalized Green's formula holds(Oden, 1976)

$$\int_\Omega (wAu - uAw)\,dx = \int_{\partial\Omega} \Gamma(u,w)\,ds, \tag{4}$$

where u and w are sufficiently smooth functions, and the bilinear form $\Gamma(u,w)$, which is a collection of boundary terms, is given by

$$\Gamma(u,w) = \sum_{i=0}^{m-1}(B_i u C_{\bar{i}} w - C_i u B_{\bar{i}} w), \tag{5}$$

where $\bar{i}$ denotes m-1-i.

An elliptic boundary value probem is stated in terms of these operators as

$$Au(x) = f(x), \quad x \in \Omega, \tag{6}$$

$$\left.\begin{array}{ll} B_i u(x) = b_i(x), & x \in \partial\Omega_i, \\ C_i u(x) = c_i(x), & x \in \partial\Omega_{m+i}, \end{array}\right\} i = 0, \cdots\cdots, m-1, \tag{7}$$

$$\text{where } \partial\Omega_i \cup \partial\Omega_{m+\bar{i}} = \partial\Omega, \quad i = 0, \cdots, m-1. \tag{8}$$

The nonhomogeneous part of the governing Eq. (6) and boundary conditions (7) are represented by the prescribed functions f(x), $b_i(x)$ and $c_i(x)$, defined on their respective domain and sub-boundaries. Note here that f(x) is included in (6), which was not considered in(Choi, 1987a).

A fundamental solution G with simple source at point x_0, is then defined for the operator A such that

$$AG(x,x_0) = \delta(x,x_0), \tag{9}$$

where the symbol δ denotes the Dirac delta function in the generalized sense. Since the function G has two arguments x and x_0, it is assumed throughout the procedure that the operator is applied to the first argument. The sequence of higher-order fundamental solutions are next defined as

$$G_i(x,x_0) = B_i\bar{G}(x_0,x), \quad i = 0, \cdots\cdots, m-1, \tag{10}$$

where $\bar{G}$ is the transpose of G, i.e., $\bar{G}(x_0, x) = G(x,x_0)$. Applying the operator A to (10), multiplying the function u(x), and integrating the resulting equation over the domain, the following property for G_i is obtained.

$$\int_\Omega u(x)AG_i(x,x_0)dx = \begin{cases} B_i u(x_0), & x_0 \in \Omega, \\ 1/2 B_i u(x_0), & x_0 \in \partial\Omega. \end{cases} \tag{11}$$

Utilizing(11), and applying the Green's formula for u which satisfies the governing Eq. (6) and G_i, the following generalized direct BIE's are obtained:

$$\alpha B_i u(x_0) + \int_{\partial\Omega}\Gamma[u(x),\ G_i(x,x_0)]ds = \int_\Omega f(x)G_i(x,x_0)dx,$$

$$\text{with } \alpha = \begin{cases} 1, & x_0 \in \Omega, \\ 1/2, & x_0 \in \partial\Omega, \end{cases} \quad i = 0, \cdots\cdots, m-1, \tag{12}$$

where ds means the integration with respect to x on $\partial\Omega$. Introduce now a sequence of sufficiently smooth functions $\{\rho_i\}_{i=0}^{m-1}$ defined on $\partial\Omega$, multiply to Eq. (12), and integrate over $\partial\Omega$ to obtain the identity

$$\sum_{i=0}^{m-1}\int_{\partial\Omega}[B_i u(x)\{1/2\rho_{i}(x) + \sum_{k=0}^{m-1}\int_{\partial\Omega}\rho_k(x_0)C_{\bar{i}}G_k(x,x_0)\,ds_0\}$$

$$-C_i u(x)\{\sum_{k=0}^{m-1}\int_{\partial\Omega}\rho_k(x_0)B_{\bar{i}}G_k(x,x_0)ds_0\}]ds$$

$$= \int_\Omega f(x)\{\sum_{i=0}^{m-1}\int_{\partial\Omega}\rho_i(x_0)G_i(x,x_0)ds_0\}dx, \tag{13}$$

where ds_0 means integration with respect to x_0 on $\partial\Omega$.

On the other hand, one can derive an indirect BIE by employing the concept of the potential theory. Introducing now a continuous distribution of simple sources with volume density $g(x_0)$ over the domain Ω, and higher order singularities with surface densities $\rho_i(x_0)$ over the boundary $\partial\Omega$, the following potential is generated:

$$w(x) = \int_\Omega g(x_0)G(x,x_0)dx_0 + \sum_{i=0}^{m-1}\int_{\partial\Omega}\rho_i(x_0)G_i(x,x_0)ds_0, \tag{14}$$

where dx_0 means the volume integral over the source point x_0, It can be shown by the property of G and G_i that w(x) satisfies the governing Eq.(6) in Ω with f replaced by g. Applying the boundary operators to w(x), and comparing with the identity (13), the expressions for the boundary terms of w in terms of ρ_i are obtained as

$$B_i w(x) = \int_\Omega g(x_0)B_iG(x,x_0)\,dx_0 + \sum_{k=0}^{m-1}\int_{\partial\Omega}\rho_k(x_0)B_iG_k(x,x_0)ds_0,$$

$$x \in \Omega \cup \partial\Omega, \tag{15}$$

$$C_i w(x) = \int_\Omega g(x_0)C_iG(x,x_0)dx_0 + \sum_{k=0}^{m-1}\int_{\partial\Omega}\rho_k(x_0)C_iG_k(x,x_0)ds_0$$

$$+1/2\rho_{\bar{i}}(x), \quad x \in \partial\Omega \tag{16}$$

$$i = 0, \cdots\cdots, m-1,$$

The term $1/2\rho_i(x)$ in (16) can be interpreted as the limiting value of the singular term appearing in the integrals of (16) as the point x approaches $\partial\Omega$. Substitution of (15) and (16) into (13) yields the following simple expression.

$$\int_{\partial\Omega}\Gamma(u,w)\,ds = \int_\Omega (fw - gu)dx. \tag{17}$$

This corresponds to the generalized Green's formula for the functions u and w which satisfy the governing equations with nonhomogeneous parts given as f and g, respectively.

3. SHAPE DESIGN SENSITIVITY OF STATIC RESPONSE

In this section, the shape design sensitivity formula is developed for the, problems with static response, using the material derivative concept and employing adjoint variable technique(Haug, 1985). The functional to be considered is general including also the domain integral which was not considered in(Choi, 1987a). Furthermore, the shape variation is described generally by considering both the normal and tangential movement of the boundary.

Consider a functional arising in shape design problems in the following form

$$\Phi = \int_\Omega h(u)dx + \int_{\partial\Omega}\Psi(B_i u, C_i u)ds, \tag{18}$$

where h and Ψ are functions which are continuously differentiable with respect to their arguments. The material derivative concept is utilized for description of the shape variation(see Fig. 1).

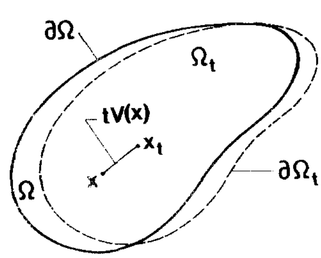

Fig. 1 Shape variation

Then, the variation of a point x in the domain is represented by a velocity field V(x) at time t=0, so that the varied point x_t in the neighborhood of t=0 is given by

$$x_t = x + tV(x) \tag{19}$$

The variation of a function u(x) with respect to a shape change is then represented by the material derivative $\dot{u}$ as follows

$$\dot{u}(x) = u'(x) + \nabla u(x)\cdot V(x), \tag{20}$$

where u′(x) is the partial derivative of u with respect to the time, that is, the variation of u with the point x held fixed. V(x) is the design velocity field at time t = 0, and ∇ denotes the gradient operator. The application of material derivative formula for Φ gives(Haug, 1985)

$$\Phi' = \int_\Omega h' dx + \int_{\partial\Omega} h V_n ds + \int_{\partial\Omega} \{\dot{\Psi} + \Psi(\text{div } V - n \cdot DVn)\} ds, \tag{21}$$

where n is the unit normal vector on $\partial\Omega$, and DV is the Jacobian matrix of V(x), that is, differentiation with respect to $x \in \mathbf{R}^n$.

Now the velocity field on the boundary can be decomposed as

$$V = V_n n + V_T, \tag{22}$$

where $V_n = V \cdot n$ is the normal component of V, and V_T is the velocity field tangential to the boundary. Note that in the present work, V_T is considered together with V_n. More general expresssions for the sensitivity formula are available by including this, and will facilitate an easy treatment of a boundary, specially when the variation is physically constrained in some directions. Substitute(22) into (21)and note that

$$\text{div } V - n \cdot DVn = V_n H + \text{div}_T V_T, \tag{23}$$

where H is the mean curvature of the boundary $\partial\Omega$ which is related to the normal vector as

$$H = \text{div } n. \tag{24}$$

The proof of (24) is given in(Zolesio, 1981). In Eq. (23), the symbol div_T denotes the surface divergence operator defined as

$$\text{div}_T V = \text{div} V - n \cdot DVn. \tag{25}$$

This is in fact a differential operator on the boundary $\partial\Omega$, involving only tangential derivatives of V.

Substitute (23) into (21), and expand h′ and $\dot{\Psi}$ in terms of their respective arguments to obtain

$$\begin{aligned} \Phi' = & \int_\Omega h_u \dot{u} dx + \sum_{i=0}^{m-1} \int_{\partial\Omega} \{\Psi_{B_i u} \overline{\dot{B_i u}} + \Psi_{C_i u} \overline{\dot{C_i u}}\} ds \\ & - \int_\Omega h_u \nabla u \cdot V dx + \int_{\partial\Omega} \{(h + \Psi H) V_n + \Psi \text{div}_T V_T\} ds, \end{aligned} \tag{26}$$

where h_u, $\Psi_{B_i u}$ and $\Psi_{C_i u}$ denote the partial derivatives.

Note that without loss of generality, the function h in the domain integral is assumed to have only function u itself as its argument. If one has some higher order derivatives of u as well, then the variations of these terms can be eliminated by integration by parts, so that only $\dot{u}$ appears in the domain integral of the resulting expression. The details will be described when concrete examples are taken(Choi, 1987b).

Now in Eq.(26), the variations of u in the domain, and $B_i u$ and $C_i u$ on the unknown parts of the boundary are related implicitly to the velocity field V through the BIE. To represent these terms with V, the identity(17) is utilized. Take the material derivative of (17) to obtain

$$\begin{aligned} & \int_{\partial\Omega} \{\Gamma(\dot{\bar{u}}, w) + \Gamma(u, \dot{\bar{w}}) + \Gamma(u, w)(V_n H + \text{div}_T V_T)\} ds \\ & = \int_\Omega \{fw' - (gu' + g'u)\} dx + \int_{\partial\Omega} (fw - gu) V_n \, ds, \end{aligned} \tag{27}$$

where $$\Gamma(\bar{u}, w) = \sum_{i=0}^{m-1} (\overline{\dot{B_i u}} C_i w - \overline{\dot{C_i u}} B_i w),$$

and $\Gamma(u, \dot{\bar{w}})$ has a similar expression. Note here that $f' = 0$ has been used. Expand now $\overline{\dot{B_i w}}$ and $\overline{\dot{C_i w}}$ in (27) to obtain

$$\left.\begin{aligned} \overline{\dot{B_i w}} &= B_i w' + \beta_i(V) w, \\ \overline{\dot{C_i w}} &= C_i w' + \gamma_i(V) w, \end{aligned}\right\} \quad i = 0, \cdots\cdots, \ m-1, \tag{28}$$

where β_i and γ_i simply represent differential operators containing V, which appear as the result of expansion. The detailed expressions are to be obtained case by case, as illustrated in(Choi, 1987b). In practice, this requires various material derivative formulas(see Zolesio, 1981). Substitute (28) into (27), noting that

$$\int_{\partial\Omega} \Gamma(u, w') ds = \int_\Omega (fw' - g'u) dx, \tag{29}$$

which corresponds to the Green's formula for u and w′. Consequently, the resulting expression for variation of (17) is given by

$$\begin{aligned} & \int_\Omega g\dot{u} \, dx + \sum_{i=0}^{m-1} \int_{\partial\Omega} \{\overline{\dot{B_i u}} C_i w - \overline{\dot{C_i u}} B_i w\} ds = \int_\Omega g \nabla u \cdot V \, dx \\ & + \int_{\partial\Omega} \{(fw - gu) V_n - \Gamma(u, w)(V_n H + \text{div}_T V_T)\} ds \\ & - \sum_{i=0}^{m-1} \int_{\partial\Omega} \{B_i u \gamma_i(V) w - C_i u \beta_i(V) w\} ds. \end{aligned} \tag{30}$$

As in(Choi, 1987a), define an adjoint system to eliminate $\dot{u}$, $\overline{\dot{B_i u}}$ and $\overline{\dot{C_i u}}$ by introducing adjoint variable u^*, such that the nonhomogeneous term f(x) in Eq.(12) is replaced by

$$f^* = h_u, \quad \text{in } \Omega, \tag{31}$$

and boundary conditions are

$$\left.\begin{aligned} B_i u^* &= -\Psi_{C_i u} \quad \text{on } \partial\Omega_i, \\ C_i u^* &= \Psi_{B_i u} \quad \text{on } \partial\Omega_{m+i}, \end{aligned}\right\} \quad i = 0, \cdots\cdots, m-1. \tag{32}$$

Substituting f^* and u^* in place of g and w in (30), and utilizing (31) and (32), one finally obtains the sensitivity formula for Φ as

$$\begin{aligned} \Phi' = & \int_{\partial\Omega} [(h + fu^* - f^* u) V_n + \{\Psi - \Gamma(u, u^*)\}(V_n H \\ & + \text{div}_T V_T)] ds - \sum_{i=0}^{m-1} \int_{\partial\Omega} \{B_i u \gamma_i(V) u^* - C_i u \beta_i(V) u^*\} ds \\ & + \sum_{i=0}^{m-1} \left\{ \int_{\partial\Omega_i} (\Psi_{B_i u} - C_i u^*) \dot{b}_i ds + \int_{\partial\Omega_{m+i}} (\Psi_{C_i u} - B_i u^*) \dot{c}_i \, ds \right\}. \end{aligned} \tag{33}$$

The present formula shows an explicit expression for the variation of Φ as a function of V. Once the solutions for primal and adjoint problems are obtained using a boundary element discretization, the sensitivity can be calculated by (33). Although comprehensive, the result derived above looks technical. For concrete problems as treated in the next paper (Choi, 1987b), many of the terms usually vanish. However, detailed calculations, especially manipulations with material derivatives of several surface variables and integration by parts on the boundary, are essential to obtain specific explicit formulas for computational applications.

4. CONCLUSIONS

It has been shown that the sensitivity analysis procedure used earlier by the authors for a boundary functional with the normal component of boundary variations, can be extended to comprehensively cover the domain integrals as well as boundary integrals. Both the tangential and normal component of the velocity field at the boundary are considered. A formal derivation of the boundary integral equation for arbitrary elliptic operator equation is presented. The resulting boundary integral identity is then used for the sensitivity analysis as a primary form of system equation. The material derivative concept and adjoint variable method are employed and their approach is shown to be well unified.

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